On the Internal Sum of Positive Monoids

Jonathan Du, Bryan Li, Nick Zhang

MIT PRIMES-USA

(Mentored by Dr. Felix Gotti)

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Outline

General Notations

Some General Notation

- $\mathbb{N}$ denotes the set of positive integers.
- $\mathbb{N}_0$ denotes the set of non-negative integers.
- $\mathbb{P}$ denotes the set of standard (positive) primes.

Positive Monoids

Definition. A monoid is a set M equipped with a binary operation "$+$" satisfying

- $(a + b) + c = a + (b + c)$ for all $a, b, c \in M$ (associativity);
- There exists an element 0 in M, often called the zero/identity element, such that $0 + a = a + 0 = a$ for all $a \in M$.
- $a + b = b + a$ for all $a, b \in M$ (commutativity).

Let M be a monoid.

- A subset N of M is called a submonoid of M if N contains the identity of M and is closed under "$+$".
- An additive submonoid of $\mathbb{R}_{\geq 0}$ is called a positive monoid.

Remark. Positive monoids are the most relevant objects in this talk.

Divisibility

Definition. Let M be a monoid.

- For $a, b \in M$, we say a (additively) divides b (or b is divisible by a) if there exists $c \in M$ such that $a + c = b$. In this case, we write

$$a \mid_M b.$$

Example. In the submonoid $N := \mathbb{N}_0 \setminus \{1\}$ of $\mathbb{N}_0$:

- $2 \mid_N 5$ because $5 = 2 + 3$ and $2, 3 \in N$, while
- $2 \nmid_N 3$ because $1 \notin N$.

Generating Sets and Finitely Generated Monoids

Definition. Let M be a monoid.

- The submonoid generated by a subset S of M, written as $\langle S \rangle$, is the smallest (under set inclusion) submonoid of M containing S.
- We call S a generating set of $\langle S \rangle$.
- We say that a monoid is finitely generated if it has a finite generating set.

Example. The additive submonoid $\mathbb{N}_0 \setminus \{1\}$ of $\mathbb{N}_0$ has $\{2, 3\}$ as a generating set, but it does not have any generating set of size 1.

Internal Sums

Definition. Let G be an abelian group, and let M_1 and M_2 be submonoids of G. The internal sum of M_1 and M_2, denoted by $M_1 + M_2$, is the submonoid of G generated by the set $M_1 \cup M_2$ (i.e., the smallest submonoid of G containing both M_1 and M_2):

$$M_1 + M_2 := \{b_1 + b_2 : b_1 \in M_1, b_2 \in M_2\}.$$

Proposition (easy to show)

If S_1 and S_2 are subsets of an abelian group G, then $\langle S_1 \rangle + \langle S_2 \rangle = \langle S_1 \cup S_2 \rangle$

Atomicity: Existence of Factorization

Definition. Let M be a monoid.

- An element $u \in M$ is invertible if there exists $u' \in M$ with $u + u' = 0$.
- A non-invertible element $a \in M$ is an atom if there does not exist non-invertibles $b, c \in M$ such that $b + c = a$. The set of atoms of M is denoted by $\mathcal{A}(M)$.
- We call an element of M atomic if it is invertible or is in the submonoid generated by $\mathcal{A}(M)$.
- The monoid M is atomic if every element of M is atomic.

Examples

- A finitely generated monoid is atomic.
- The monoid $\left\langle \frac{1}{p} : p \in \mathbb{P} \right\rangle$ is atomic because one can show that each $\frac{1}{p}$ is an atom, which means the entire monoid is generated by atoms.

Theorem (D-L-Z, 2024)

Let M and N be positive monoids such that N is finitely generated and M is atomic, then $M + N$ is atomic.

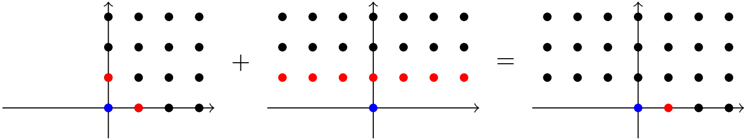
Preservation of Atomicity under Internal Sum

Remark. The internal sum of a general atomic monoid and a finitely generated monoid is not necessarily atomic.

Example. There exists an atomic monoid whose internal sum with a finitely generated monoid is not atomic.

- Set $M_1 := \langle (1,0),(0,1) \rangle$, a finitely generated monoid.
- Set $M_2 := \langle (n,1) : n \in \mathbb{Z} \rangle$, an atomic monoid.

The only atom of the monoid $M := M_1 + M_2$ is $(1,0)$.



Remark. The internal sum of two atomic positive monoids may not even have any atoms (although the construction is somehow technical).

The Ascending Chain Condition on Principal Ideals (ACCP)

Definition. Let M be a monoid.

- A principal ideal of M is a subset of M of the form $b + M$ for some $b \in M$.
- An ascending chain of subsets of M is a sequence of subsets $S_1, S_2, S_3, \dots \subset M$ such that $S_1 \subset S_2 \subset S_3 \subset \cdots$.
- An ascending chain is said to stabilize if there exists $N \in \mathbb{N}$ such that $S_N = S_i$ for all $i \geq N$.
- The monoid M satisfies the ACCP if, for all $b \in M$, every ascending chain of principal ideals starting at $b + M$ stabilizes.

Remark. All ACCP monoids are atomic.

Theorem (Geroldinger-Gotti, 2024)

Let M and N be positive monoids such that N is finitely generated. If M satisfies the ACCP, then $M + N$ satisfies the ACCP.

Preservation of ACCP under Internal Sum

Example: There exist two ACCP positive monoids whose internal sum does not satisfy the ACCP.

1. Set $M_1 := \{0\} \cup \mathbb{Q}_{\geq 1}$.
2. Suppose M_1 is not ACCP; then there exists an infinite, strictly ascending chain of principal ideals

$$a_1 + M_1 \subsetneq a_2 + M_1 \subsetneq a_3 + M_1 \subsetneq \cdots .$$

3. Note that $a_2 \mid_{M_1} a_1 \implies a_1 - a_2 \in M_1$. But since $a_1 - a_2 \neq 0$, we must have $a_1 - a_2 \geq 1$.
4. Similarly, $a_2 - a_3 \geq 1$, and so on. Thus, $a_1 - a_{n+1} \geq n$ for all $n \in \mathbb{N}$. Since a_n is always nonnegative, this is impossible.

Preservation of ACCP under Internal Sum

5. Set
$$M_2 := \left\langle \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \right\rangle,$$
where p_n is the n-th prime.

6. It is possible to show that M_2 is ACCP, and in fact, a finite factorization monoid (a fact that will be defined and demonstrated later).

Preservation of ACCP under Internal Sum

7. Let $M := M_1 + M_2$. We will show that M does not satisfy the ACCP; in fact, it is not atomic.

8. $\frac{p_n+1}{p_n^2}$ grows arbitrarily close to zero for large $n \in \mathbb{N}$.

9. Thus, any element $b \in M$ with $b > 1$ is divisible in M by $\frac{p_n+1}{p_n^2}$ for some large $n \in \mathbb{N}$. Hence b is not an atom.

10. Hence,
$$\mathcal{A}(M) \subseteq \{1\} \cup \left\{ \frac{p_n + 1}{p_n^2} : n \in \mathbb{N} \right\}.$$

11. Note that the sum of atoms of M cannot have the cube of any prime in its denominator.

12. Thus, the element $\frac{9}{8} \in M$ is not atomic.

13. Hence, M is not an atomic monoid, thus is not ACCP.

Factorization and the BFM Property

Definitions. Let M be a monoid.

1. Let r be an element of M. An (additive) factorization of r is defined as a formal sum of finitely many atoms of M equaling r, and define the length of a factorization of $r \in M$ is its number of atoms (counting repetitions).
2. The monoid M is called a bounded factorization monoid (BFM) if for all noninvertible elements $r \in M$ there is an upper bound on the length of all factorizations of r.

Example. The monoid $M_1 := \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM.

Remark. Every positive monoid M such that zero is not a limit point of $M \setminus \{0\}$ is a BFM.

Theorem (D-L-Z, 2024)

Let M and N be positive monoids such that N is finitely generated. Then if M is a BFM, then $M + N$ is a BFM.

The BFM Property under Internal Sum

Example. Let $M_1 := \{0\} \cup \mathbb{Q}_{\geq 1}$, and let $M_2 := \left\langle \frac{p_n+1}{p_n^2} : n \in \mathbb{N} \right\rangle$, where p_n is the n-th prime.

1. Consider any element $r \in M_2$. We can set $r = k + \frac{a}{b}$, where $k \in \mathbb{N}$ and $0 \leq a < b$.
2. Only a finite number of primes p_n satisfy $p_n + 1 \leq k$, and only a finite number of primes p_n satisfy $p_n \mid b$.
3. In order for $\frac{p_n+1}{p_n^2}$ to divide $r = k + \frac{a}{b}$, at least one of the above statements must be true.
4. Only finitely many atoms of M_2 can divide r, so there exists a minimal divisor d of r.
5. Thus the length of a factorization of r is at most $\frac{r}{d}$, so M_2 is a BFM.

Finite Factorization Monoids

Definition. A monoid M is a finite factorization monoid (FFM) if every element $r \in M$ is only divisible by finitely many elements of M (up to associates).

Remarks

1. Every FFM is a BFM.
2. Every finitely generated monoid is an FFM.

Example. $M_2 := \left\langle \frac{p_n+1}{p_n^2} : n \in \mathbb{N} \right\rangle$ is an FFM.

Theorem (D-L-Z, 2024)

Let M and N be positive monoids such that N is finitely generated. Then if M is an FFM, then $M + N$ is an FFM.

The FFM Property under Internal Sum

Example. Let $M_1 := \left\langle \frac{p-1}{p} : p \in \mathbb{P}_{\geq 5} \right\rangle$, and let $M_2 := \left\langle \frac{p+1}{p} : p \in \mathbb{P}_{\geq 5} \right\rangle$. M_1 and M_2 are both FFMs, but their internal sum $M := M_1 + M_2$ is not an FFM.

1. Consider any element $r \in M_1$. Once again, we can write $r = n + \frac{a}{b}$, where $n \in \mathbb{N}$ and $a < b$.
2. Only finitely many primes p satisfy $p - 1 \leq n$ or $p \mid b$.
3. Only finitely many atoms $\frac{p-1}{p} \in M_1$ can satisfy $\frac{p-1}{p} \mid_{M_1} r$.
4. This implies that only finitely many elements $d \in M_1$ can divide r.
5. M_2 can similarly be shown to be an FFM.
6. However, their internal sum $M = M_1 + M_2$ contains the element 2, which can be written as $2 = \frac{p-1}{p} + \frac{p+1}{p}$ for all $p \in \mathbb{P}_{\geq 5}$, so M is not an FFM.

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End of Presentation

THANK YOU!